Linear combination of independent normal random variables

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Let $Z_1, Z_2 \sim \mathcal{N}(0,1)$ be independent, and let $X = aZ_1 + bZ_2$ where a, b are scalars. We wish to find the distribution of X.

We know that for independent X_1, X_2, if $X = f(X_1, X_2)$ is invertible so that $X_2 = g(X_1, X)$, then:

$$p_X(x) = \int_{x_1} p_{X_1}(x_1) p_{X_2}(g(x_1, x)) \cdot \left| \frac{\partial f}{\partial x_2} \right|^{-1}_{g(x_1,x)} dx_1 \tag{1}$$

Hence,

$$\begin{aligned} p_X(x) &= \int_{z_1=-\infty}^{\infty} p_{Z_1}(z_1) p_{Z_2}\left(\frac{x - az_1}{b}\right) \cdot \left|\frac{1}{b}\right| dz_1 \\ &= \frac{1}{|b|} \int_{z_1=-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-z_1^2}{2}\right) \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-(x-az_1)^2}{2b^2}\right) dz_1 \\ &= \frac{1}{2\pi|b|} \int_{z_1=-\infty}^{\infty} \exp\left(\frac{-1}{2}\left(z_1^2 + \frac{(x-az_1)^2}{b^2}\right)\right) dz_1 \end{aligned} \tag{2}$$

Now consider,

$$\begin{aligned} Q &= z_1^2 + \frac{(x-az_1)^2}{b^2} \\ &= \left(1 + \frac{a^2}{b^2}\right) z_1^2 - 2\frac{az_1 x}{b^2} + \frac{x^2}{b^2} \\ &= \left(\frac{a^2+b^2}{b^2}\right) z_1^2 - 2\frac{az_1 x}{b^2} + \frac{x^2}{b^2} \\ &= \frac{1}{b^2}\left(\left(a^2+b^2\right) z_1^2 - 2az_1 x + x^2\right) \end{aligned} \tag{3}$$

We can now complete the square for the first two terms. The last term is independent of z_1 so its exponential will come out of the integral. Thus,

$$\begin{aligned} Q &= \frac{1}{b^2}\left(\left(a^2+b^2\right) z_1^2 - 2az_1 x + \frac{a^2x^2}{a^2+b^2} + x^2 - \frac{a^2x^2}{a^2+b^2}\right) \\ &= \frac{1}{b^2}\left(\left(\sqrt{a^2+b^2} z_1 - \frac{ax}{\sqrt{a^2+b^2}}\right)^2 + \frac{b^2x^2}{a^2+b^2}\right) \end{aligned} \tag{4}$$

Using this in Eq(2) we get:

$$p_X(x) = \frac{1}{2\pi|b|} \exp\left(\frac{-x^2}{2(a^2+b^2)}\right) \int_{z_1=-\infty}^{\infty} \exp\left(\frac{-1}{2b^2}\left(\sqrt{a^2+b^2} z_1 - \frac{ax}{\sqrt{a^2+b^2}}\right)^2\right) dz_1 \tag{5}$$

Let

$$t = \sqrt{a^2+b^2}z_1 - \frac{ax}{\sqrt{a^2+b^2}}$$

Then $dt = \sqrt{a^2+b^2}dz_1$. Also, limits won't change as they are infinite. Hence,

$$\begin{aligned} p_X(x) &= \frac{1}{2\pi|b|}\exp\left(\frac{-x^2}{2(a^2+b^2)}\right)\int_{t=-\infty}^{\infty}\exp\left(\frac{-t^2}{2b^2}\right)\cdot\frac{1}{\sqrt{a^2+b^2}}dt \\ &= \frac{1}{2\pi|b|}\cdot\frac{1}{\sqrt{a^2+b^2}}\exp\left(\frac{-x^2}{2(a^2+b^2)}\right)\int_{t=-\infty}^{\infty}\exp\left(\frac{-t^2}{2b^2}\right)dt \\ &= \frac{1}{2\pi|b|}\cdot\frac{1}{\sqrt{a^2+b^2}}\exp\left(\frac{-x^2}{2(a^2+b^2)}\right)2\int_{t=0}^{\infty}\exp\left(\frac{-t^2}{2b^2}\right)dt \end{aligned} \tag{6}$$

In the last step we used the fact that the integral is even. Also, we already know that:

$$\int_{x=0}^{\infty} x^m e^{-\beta x^n}dx = \frac{1}{n}\frac{\Gamma(\frac{m+1}{n})}{\beta^{\frac{m+1}{n}}} \tag{7}$$

In the last integral in Eq(6), $m=0$, $n=2$, and $\beta = 1/(2b^2)$. Hence:

$$p_X(x) = \frac{1}{2\pi|b|}\cdot\frac{1}{\sqrt{a^2+b^2}}\exp\left(\frac{-x^2}{2(a^2+b^2)}\right)2\cdot\frac{1}{2}\frac{\Gamma(1/2)}{(1/\sqrt{2}|b|)} \tag{8}$$

Since $\Gamma(1/2) = \sqrt{\pi}$, this gives us:

$$\boxed{p_X(x) = \frac{1}{\sqrt{2\pi(a^2+b^2)}}\exp\left(\frac{-x^2}{2(a^2+b^2)}\right)} \tag{9}$$

This means that $X = aZ_1 + bZ_2$ is also normally distributed with mean 0 and variance a^2+b^2. We can use this to easily derive the density of independent general normal random variables. Let $X_1 = \mathcal{N}(\mu_1, \sigma_1^2)$ and $X_2 = \mathcal{N}(\mu_2, \sigma_2^2)$ be independent, and let $X = aX_1 + bX_2$. We can write:

$$\begin{aligned} X &= aX_1 + bX_2 \\ &= a(X_1-\mu_1) + b(X_2-\mu_2) + (a\mu_1 + b\mu_2) \\ &= a\sigma_1\left(\frac{X_1-\mu_1}{\sigma_1}\right) + b\sigma_2\left(\frac{X_2-\mu_2}{\sigma_2}\right) + (a\mu_1+b\mu_2) \\ &= a\sigma_1 Z_1 + b\sigma_2 Z_2 + (a\mu_1+b\mu_2) \end{aligned} \tag{10}$$

This is also a linear combination of standard normals with coefficients $a\sigma_1$ and $b\sigma_2$ shifted by $a\mu_1 + b\mu_2$. Using the previous result, we finally have:

$$\boxed{X \sim \mathcal{N}\left(a\mu_1 + b\mu_2, a^2\sigma_1^2 + b^2\sigma_2^2\right)} \tag{11}$$

We can now use the induction to generalize this to n indepedent normal random variables. Thus, if $X_1, X_2, \cdots, X_n$ are independent with $X_i \sim \mathcal{N}(\mu_i, \sigma_i^2)$, and if $c_1, c_2, \cdots, c_n$ are scalars, then:

$$\boxed{X = \sum_{i=1}^{n} c_i X_i \sim \mathcal{N}\left(\sum_{i=1}^{n} c_i\mu_i, \sum_{i=1}^{n} c_i^2\sigma_i^2\right)} \tag{12}$$

Two points are worth being remembered.

- When $X_1, X_2, \cdots, X_n$ are independent normal, then $\overline{X}_n$ is normal, because it is just a linear combination of normals with same mean and variance, with each $c_i = 1/n$.

- Second, the normal random variables $X_1, X_2, \cdots, X_n$ need not necessarily be independent, but only have to be components of multivariate normal. We have not yet studied this topic yet, but when independence holds, then these are trivially components of a multivariate normal whose density is the product of their densities. Conversely, if *every* linear combination of $X_1, X_2, \cdots, X_n$ is normal, then they form multivariate normal.